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# Representations of global supersymmetries for all $N$

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**Abstract.** We present the global supersymmetry rules for the fundamental representation of $N$-extended supersymmetry for all $N$ and an associated invariant quadratic Lagrangian. This is achieved by the use of a suitable set of basis functions defined in terms of the spinor generators acting on the vacuum state of the representation. These basis functions allow the action of the generators on them to be obtained explicitly, as well as their internal symmetry properties. Specific examples are given of this for $N = 3$, 4, 6 and 8.

## 1. Introduction

In order to obtain a full quantum field theory for $N$-extended supergravity with $N \leq 8$ in explicitly superfield-theoretic or locally supersymmetric form it is necessary to have the auxiliary field structures. These latter are also of greatest importance in obtaining the torsion constraints in a superspace formulation in a differential geometric framework. These auxiliary fields are now known for $N = 1$ (Stelle and West 1978, Ferrara and van Nieuwenhuizen 1978, Sohnius and West 1981) and 2 (Fradkin and Vasiliev 1979, de Wit and van Holten 1979). It has been shown that there is a barrier to the usual constructions at $N = 3$ (Rivelles and Taylor 1981, 1982c, Taylor 1982a) where representations with central charge are essential in order to circumvent the 'no-go' theorems.

A method has been developed recently (Taylor 1982b, c, Taylor and Rivelles 1982, Rivelles and Taylor 1982d) for obtaining the auxiliary fields at the linearised level by discovering those irreps of the $N$-extended supersymmetry algebra, $S_N$, which may be combined through field redefinition rules to produce linearised $N$-extended supergravity with no non-local terms in the SUSY transformation laws for the redefined fields. These latter are the physical fields known for $N$ up to 8 and the auxiliary fields yet to be discovered for $N \geq 3$. A similar approach can be used to construct $N = 4$ SYM where again no auxiliary field structure is presently known.

The method described above has been used to discover two new non-minimal 20+20 formulations of $N = 1$ supergravity at the linearised level (Rivelles and Taylor 1982a) and prove that there are no other auxiliary field sets for this value of $N$. It has also been used to discover (Rivelles and Taylor 1982b) a new 40+40 set of auxiliary fields for $N = 2$ supergravity at the linear level, in this case containing a central charge on the auxiliary multiplets.

In order to use this method to construct linearised off-shell supergravity for higher $N$ it is necessary to have the irreducible representations (irreps) of the associated algebra, $S_N$. We present the transformation laws for the component fields and an associated invariant quadratic Lagrangian for the fundamental superspin $Y = 0$ and

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SU($N$) singlet irrep. This is done for general $N$ by means of basis functions of the fundamental irrep which were introduced earlier for $N = 1$ and 2 (Jarvis 1976, Taylor 1980a, b). These functions form a halfway house between the full superfield approach and a component approach and, as such, are of value for providing a bridge between these two extremes.

In the next section we define these basis functions for any $N$ and obtain their transformations under global SUSY in § 3. The transformation rules for the associated component fields are deduced in § 4 along with appropriate reality conditions. These conditions were introduced originally for $N = 2$ SUSY (Firth and Jenkins 1975) and extended to general $N$ in terms of superprojectors (Siegel and Gates 1981). In § 5 we give the specific transformation laws and Lagrangians for the interesting cases of $N = 3, 4, 6, 8$. The case $N = 4$ has already been published (de Wit 1981), the other cases appearing here for the first time to our knowledge. In a final section we discuss the relevance of these results.

## 2. Basis functions

In order to specify our notation, and also to define the basis functions of a particular irreducible representation of the SUSY algebra $S_N$, we consider the SUSY generators $(S^i_{\alpha+}, S_{\alpha-i})$ in chiral notation with $(S^i_{\alpha+})^* = S_{\alpha-i}$. The algebra satisfied by these generators, in the absence of central charges, will be

$$[S^i_{\alpha+}, S_{\beta-j}]_+ = -2(\not{p}\eta)_{\alpha+\beta-}\delta^i{}_j \qquad [S^i_{\alpha+}, S^j_{\beta+}]_+ = 0 \tag{2.1}$$

with $\eta$ the charge conjugation matrix and the generators $P_\mu$, $J_{\mu\nu}$ have standard commutators. The representation theory of $S_N$ (Taylor 1980a, b, 1982b, c, Taylor and Rivelles 1982, Rivelles and Taylor 1982d, Jarvis 1976, Pickup and Taylor 1981, Pickup unpublished, Rittenberg and Sokatchev 1981, Ferrara *et al* 1981, Ferrara and Savoy 1982) shows that each irrep may be classified by various Casimirs constructed from the covariant operators $(D^i_{\alpha+}, D_{\alpha-i})$ with

$$\begin{aligned}&[S^i_{\alpha+}, D^j_{\alpha+}]_+ = [S^i_{\alpha+}, D_{\beta-j}]_+ = 0 \text{ etc}\\ &[D^i_{\alpha+}, D_{\beta-j}]_+ = +2(\not{p}\eta)_{\alpha+\beta-}\delta^i{}_j \qquad [D^i_{\alpha+}, D^j_{\beta+}]_+ = 0.\end{aligned} \tag{2.2}$$

The extension of the Pauli–Lubanski vector for the Poincaré group to $S_N$ gives as a Casimir the superspin $Y$, with values $0, \frac{1}{2}, \ldots$ . We may also extend the SU($N$) operators acting as automorphisms of $S_N$ so as to commute with $S_N$; the associated Casimirs correspond to those of irreps of SU($N$). Each irrep, with again $Y$ and SU($N$) classification, has Lorentz spin content classified by USp($2N$), each spin value having states classified by SU($N$). This irrep can be constructed by applying the generators $S_{\alpha-i}$ to a suitable vacuum state with Lorentz and SU($N$) transformation laws of the irrep. The states may then be reduced by the usual rules of direct product decomposition of irreps of the Lorentz group and SU($N$).

We will concentrate our attention on the $Y = 0$ singlet SU($N$) irrep (the fundamental irrep), since all other irreps may be obtained by addition of suitable Lorentz and SU($N$) indices to all states of the fundamental irrep. We proceed by defining the vacuum state of the irrep as a differential operator on component functions as

$$|0\rangle = \prod_{i=1}^{N} (\bar{S}^i_+ S^i_+) \cdot 1. \tag{2.3}$$

By (2.3) we mean that we consider $S^i_+$ as the differential operator

$$S^i_{\alpha+} = \mathrm{i}(\partial/\partial\bar{\theta}_{\alpha+i} + (\not{p}\theta^i_-)_{\alpha+}) \tag{2.4}$$

acting on the constant function of $\theta$; the action of $\not{p}$ is thus non-trivial. That $|0\rangle$ in (2.3) is the chiral $Y=0$ vacuum is clear from the fact that we may replace $D^i_{\alpha+}$ by $S^i_{\alpha+}$ when acting on 1, so that

$$D^i_{\alpha+}|0\rangle = \prod_{i=1}^{N} (\bar{S}^i_+ S^i_+) S^j_{\alpha+} \cdot 1 = 0 \tag{2.5}$$

which is the condition for the chiral vacuum state. Equation (2.5) will also be satisfied by all states constructed from (2.3) by application of products of $S_{\alpha-j}$. We must decompose these according to irreps of the Lorentz and SU($N$) groups, and so, by standard arguments, have the complete set of basis functions

$$e_{\alpha_1-\ldots\alpha_l-\,k_1\ldots k_m}{}^{r_1\ldots r_{N-l-m}} = \varepsilon^{i_1\ldots i_l j_1\ldots j_m r_1\ldots r_{N-l-m}} \prod_{u=1}^{m} (\bar{S}_{-j_u} S_{-k_u}) \prod_{t=1}^{l} S_{\alpha_t - i_t}|0\rangle. \tag{2.6}$$

We then get the superfield expansion

$$\phi = \sum_{l,m} e_{\alpha_1-\ldots\alpha_l-\,k_1\ldots k_m}{}^{r_1\ldots r_{N-l-m}} A^{\alpha_1-\ldots\alpha_l-\,k_1\ldots k_m}{}_{r_1\ldots r_{N-l-m}} \tag{2.7}$$

where we take the field components, $A^{\alpha_1-\ldots\alpha_l-\,k_1\ldots k_m}{}_{r_1\ldots r_{N-l-m}}$, to have the same structure as the basis functions i.e. symmetry in the spinor components and asymmetry in the two sets (upstairs, downstairs) of internal indices with a $\delta$-traceless condition between the two. This gives the field component the same characteristics as the basis function and so it carries the same Lorentz and SU($N$) labels.

## 3. Transformation of basis functions

In order to obtain the SUSY transformation rules of the component functions of (2.7), we need to obtain those of the basis functions (2.6). We therefore consider the effect of acting with an extra $S_{\beta-i}$ on the basis functions (2.6). The method of direct product multiplication tells us that this would add an extra box to the Young tableau for the SU($N$) irrep of the basis function. Hence, labelling the basis function by the number of its downstairs indices, $(l, m)$, we see that under the action of $S_{\beta-j}$,

$$(l, m) \to (l+1, m) + (l-1, m+1).$$

Bearing in mind that the basis functions are $\delta$-traceless on the $k$'s and $r$'s, we get

$$\begin{aligned} S_{\beta-i} e_{\alpha_1-\ldots\alpha_l-\,k_1\ldots k_m}{}^{r_1\ldots r_{N-l-m}} &= a\delta_i^{[r_1} e_{\beta-\alpha_1-\ldots\alpha_l-\,k_1\ldots k_m}{}^{r_2\ldots r_{N-l-m}]} + b\delta_{[i}^{[r_1} e_{\beta-\alpha_1-\ldots\,k_1\ldots k_m]}{}^{r_2\ldots]} \\ &\quad + c\eta_{\beta-(\alpha_1-} e_{\alpha_2-\ldots\alpha_l-)\,ik_1\ldots k_m}{}^{r_1\ldots r_{N-l-m}} \end{aligned} \tag{3.1}$$

where here we use the convention of no dividing factors in the symmetrisation or antisymmetrisation.

The $\delta$-traceless condition then gives us

$$b = [(l+1)m']^{-1} a. \tag{3.2}$$

Contracting with $\delta^i_{r_1}$ and $\eta^{\beta-\alpha_1-}$ in turn, both on the RHS of (3.1) and explicitly on the LHS, using (2.6), we get the values

$$a=\frac{(-1)^{l+m}(N-l-m)}{(2+l+m)(N-l-m)!},\qquad b=\frac{(-1)^{l+m}(N-l-m)}{(2+l+m)(N-l-m)!\,m!\,(l+1)},$$

$$c=\frac{(-1)^{l}l}{(l+1)!}. \tag{3.3}$$

If, instead, we were to act with $S^{i}_{\beta+}$ on the basis functions (2.6) we would get, using (2.1) to pull the $S_+$ through onto the $S_+$'s in $|0\rangle$,

$$\begin{aligned}S^{i}_{\beta+}e_{\alpha_1-\ldots\alpha_l-}{}^{r_1\ldots r_{N-l-m}}_{k_1\ldots k_m}
&=\frac{(-1)^{l+m}l}{l!}2(\not p\eta)_{\beta+(\alpha_1-}e_{\alpha_2-\ldots)}{}^{ir_1\ldots r_{N-l-m}}_{k_1\ldots k_m}\\
&\quad+\frac{(-1)^{l+1}m}{m!}2(\not p)_{\beta+}{}^{\rho-}\delta^{i}_{[k_1}e_{\rho-\alpha_1-\ldots}{}^{r_1\ldots}_{k_2\ldots k_m]}\\
&\quad+\frac{(-1)^{m}m}{m!}2(\not p)_{\beta+}{}^{\rho-}S_{\rho-[k_1}e_{\alpha_1-\ldots}{}^{r_1\ldots r_{N-l-m}}_{k_2\ldots k_m]}.\end{aligned} \tag{3.4}$$

The last term is then given by (3.1), giving us the transformation rule

$$\begin{aligned}S^{i}_{\beta+}e_{\alpha_1-\ldots\alpha_l-}{}^{r_1\ldots r_{N-l-m}}_{k_1\ldots k_m}
&=a'2(\not p)_{\beta+}{}^{\gamma-}\delta^{i}_{[k_1}e_{\gamma-\alpha_1-\ldots}{}^{r_1\ldots r_{N-l-m}}_{k_2\ldots k_m]}\\
&\quad+b'2(\not p)_{\beta+}{}^{\gamma-}\delta^{[i}_{[k_1}e_{\gamma-\alpha_1-\ldots\alpha_l-}{}^{r_1\ldots r_{N-l-m}]}_{k_2\ldots k_m]}\\
&\quad+c'2(\not p\eta)_{\beta+(\alpha_1-}e_{\alpha_2-\ldots\alpha_l-)}{}^{ir_1\ldots r_{N-l-m}}_{k_1\ldots k_m}\end{aligned} \tag{3.5}$$

where

$$a'=\frac{(-1)^{l+1}m}{m!},\qquad b'=\frac{(-1)^{l+1}m}{m!\,(N-m-l)!\,(l+1)},\qquad c'=\frac{(-1)^{l+m}l(1+l+m)}{(l+1)!}. \tag{3.5a}$$

In the special case that $l=0$, $m=N/2$ (for even $N$), there is a double $\varepsilon$ condition on $e^{r_1\ldots r_{N/2}}_{k_1\ldots k_{N/2}}$ of form

$$e^{r_1\ldots r_{N/2}}_{k_1\ldots k_{N/2}}=\frac{(-1)^{N/2}}{[(N/2)!]^2}\varepsilon^{r_1\ldots r_{N/2}\,s_1\ldots s_{N/2}}\varepsilon_{k_1\ldots k_{N/2}\,l_1\ldots l_{N/2}}e^{l_1\ldots l_{N/2}}_{s_1\ldots s_{N/2}}.$$

If this is applied to (3.1) we see that we may combine the first and second terms on the RHS, whilst the third term is zero due to $l=0$. The same arguments as above lead to

$$\begin{aligned}S_{\beta-i}e^{r_1\ldots r_{N/2}}_{k_1\ldots k_{N/2}}
&=\tfrac{1}{2}[a+(\tfrac{1}{2}N)!\,b](\delta^{[r_1}_{i}e_{\beta-}{}^{r_2\ldots r_{N/2}]}_{k_1\ldots k_{N/2}}\\
&\quad+\frac{(-1)^{N/2}}{[(\frac{1}{2}N)!]^2}\varepsilon^{r_1\ldots r_{N/2}\,s_1\ldots s_{N/2}}\varepsilon_{k_1\ldots k_{N/2}\,l_1\ldots l_{N/2}}\delta^{[l_1}_{i}e_{\beta-}{}^{l_2\ldots l_{N/2}]}_{s_1\ldots s_{N/2}})\end{aligned} \tag{3.1a}$$

and a similar expression for $s^{i}_{\beta+}e^{r_1\ldots r_{N/2}}_{k_1\ldots k_{N/2}}$; the values of $a$, $b$, $a'$ and $b'$ entering in these formulae are unchanged from (3.3) and (3.5$a$).

In general, we may impose a reality condition on the basis functions in the case of even $N$. These transformation rules ((3.1), (3.3) and (3.5)) then enable us to evaluate this reality condition explicitly for the basis function in this case. Under such circumstances we can relate basis functions of opposite chirality by

$$Fe_{\alpha_1-\ldots\alpha_l-\,k_1\ldots k_m}^{\quad r_1\ldots r_{N-l-m}} = C(N,l,m)2^l \prod_{n=1}^{l} \not{p}_{\alpha_n-}{}^{\beta_n+} e_{\beta_1+\ldots\beta_l+\,k_1\ldots k_m}^{\quad r_1\ldots r_{N-l-m}} \tag{3.6}$$

where $F = \Pi D_+D_+ \Pi D_-D_-$.

Since $S_{\gamma-i}$ commutes with $F$ we can act with it on either side of (3.6), and use (2.1), (3.3) and (3.5) to give

$$\begin{aligned} F(a\delta_i^{[r_1} e_{\gamma-\alpha_1-\ldots\,k_1\ldots k_m}^{\quad r_2\ldots]} + b\delta_{[i}^{[r_1} e_{\gamma-\alpha_1-\ldots\,k_1\ldots k_m]}^{\quad r_2\ldots r_{N-l-m}]} + c\eta_{\gamma-(\alpha_1-}e_{\alpha_2-\ldots)\,ik_1\ldots k_m}^{\quad r_1\ldots r_{N-l-m}}) \\ = C(N,l,m)2^l \prod_{t=1}^{l} \not{p}_{\alpha_t-}{}^{\beta_t+}(a''2(\not{p})_{\gamma-}{}^{\rho+}\delta_i^{[r_1}e_{\rho+\beta_1+\ldots\beta_l+\,k_1\ldots k_m}^{\quad r_2\ldots r_{N-l-m}]} \\ + b''2(\not{p})_{\gamma-}{}^{\rho+}\delta_{[i}^{[r_1}e_{\rho+\beta_1+\ldots\,k_1\ldots k_m]}^{\quad r_2\ldots r_{N-l-m}]} + c''2(\not{p}\eta)_{\gamma-(\beta_1+}e_{\beta_2+\ldots\beta_l+)\,ik_1\ldots k_m}^{\quad r_1\ldots r_{N-l-m}}) \end{aligned} \tag{3.7}$$

where

$$a'' = \frac{(-1)^{l+1}(N-l-m)}{(N-l-m)!}, \qquad b'' = \frac{(-1)^{l+1}(N-l-m)}{(N-l-m)!\,m!\,(l+1)},$$

$$c'' = \frac{(-1)^{N-m}l(N-m+1)}{(l+1)!}.$$

These coefficients are the SU($N$)-conjugate versions of $a'$, $b'$, $c'$.

By applying (3.6) and equating terms we get

$$aC(N,l+1,m) = a''C(N,l,m) \tag{3.8a}$$

$$bC(N,l+1,m) = b''C(N,l,m) \tag{3.8b}$$

$$cC(N,l-1,m+1) = -4p^2c''C(N,l,m). \tag{3.8c}$$

On substitution, the first two are equivalent and we get

$$\begin{aligned} C(N,l+1,m) &= (-1)^{m+1}(2+l+m)C(N,l,m) \\ C(N,l-1,m+1) &= (-1)^{N-l-m+1}(4p^2)(N-m+1)C(N,l,m) \end{aligned} \tag{3.9}$$

where these are properly defined, i.e. $0 \leq l$, $0 \leq m$, $0 \leq N-l-m$. These equations are also true for $n = N/2$, although the derivation is slightly different.

Solving (3.9) recursively gives us

$$C(N,l,m) = -(-1)^{(l+1)(m+1)}\frac{(1+l+m)!\,(N+1)!}{(N-m+1)!}(4p^2)^m C(N,0,0). \tag{3.10}$$

But $F^2 = (16p^2)^N$, so applying $F$ twice gives

$$\begin{aligned} &F^2 e_{\alpha_1\,\alpha_l-\,k_1\ldots k_m}^{\quad r_1\ldots r_{N-l-m}} \\ &\quad = C(N,l,m)C(N,l,N-l-m)(4p^2)^l e_{\alpha_1-\ldots\alpha_l-\,k_1\ldots k_m}^{\quad r_1\ldots r_{N-l-m}} \\ &\quad = (-1)^{(l+1)(m+1+N-l-m+1)}((N+1)!\,C(N,0,0))^2(4p^2)^N e_{\alpha_1-\ldots\,k_1\ldots k_m}^{\quad r_1\ldots r_{N-l-m}} \end{aligned}$$

giving

$$\begin{aligned} C(N, 0, 0) &= 2^N/(N+1)! \\ C(N, l, m) &= -(-1)^{(l+1)(m+1)}(4p^2)^m 2^N (1+l+m)!/(N-m+1)!. \end{aligned} \tag{3.11}$$

This gives us our reality condition between positive and negative chirality basis functions. This will enable us to obtain similar conditions on the field components in § 4.

## 4. Transformation of components

The usual definition of the SUSY transformations $\delta A$ of the component functions $A$ of (2.7) is

$$\delta\phi = (\bar{\varepsilon}S)\phi = \sum_{l,m} (\bar{\varepsilon}S)eA = \sum_{l,m} e\delta A \tag{4.1}$$

where we have suppressed the indices for compactness. From § 3 we know the effect of $\delta$ on the basis functions:

$$\begin{aligned} &(\bar{\varepsilon}S)e_{\alpha_1-\dots\alpha_l-}{}^{r_1\dots r_{N-l-m}}_{k_1\dots k_m} \\ &\quad = (\bar{\varepsilon}^{i\beta-}S_{\beta-i} + \bar{\varepsilon}_i^{\beta+}S^i_{\beta+})e_{\alpha_1-\dots\alpha_l-}{}^{r_1\dots r_{N-l-m}}_{k_1\dots k_m} \\ &\quad = a\bar{\varepsilon}^{[r_1\beta-}e_{\beta-\alpha_1-\dots}{}^{r_2\dots]}_{k_1\dots k_m} + b\bar{\varepsilon}^{i\beta-}\delta^{[r_1}_{[i}e_{\beta-\alpha_1-\dots}{}^{r_2\dots r_{N-l-m}]}_{k_1\dots k_m]} \\ &\qquad + c\varepsilon^{i}{}_{(\alpha_1-}e_{\alpha_2-\dots\alpha_l-)ik_1\dots k_m}{}^{r_1\dots r_{N-l-m}} + a'2(\bar{\varepsilon}_{+[k_1}\not{p})^{\gamma-}e_{\gamma-\alpha_1-\dots\alpha_l-}{}^{r_1\dots r_{N-l-m}}_{k_2\dots k_m]} \\ &\qquad + b'(2\bar{\varepsilon}_{+i}\not{p})^{\gamma-}\delta^{[i}_{[k_1}e_{\gamma-\alpha_1-\dots\alpha_l-}{}^{r_1\dots r_{N-l-m}]}_{k_2\dots k_m]} + c'2(\bar{\varepsilon}_{+i}\not{p}\eta)_{(\alpha_1-}e_{\alpha_1-\dots\alpha_l-)}{}^{ir_1\dots r_{N-l-m}}_{k_1\dots k_m} \end{aligned} \tag{4.2}$$

where $a, b, c, a', b', c'$ are given in (3.3) and (3.5$a$).

Contracting with $A^{\alpha_1-\dots\alpha_l-k_1\dots k_m}{}_{r_1\dots r_{N-l-m}}$ and remembering that $\delta^r_k A = 0$, we get

$$\begin{aligned} (\bar{\varepsilon}S)(eA) = {} & \frac{(a+m!b)(N-l-m)!}{(l+1)!}\bar{\varepsilon}^{r_1(\beta-}e_{\beta-\alpha_1-\dots\alpha_l-}{}^{r_2\dots r_{N-l-m}}_{k_1\dots k_m}A^{\alpha_1-\dots\alpha_l-)k_1\dots k_m}{}_{r_1\dots r_{N-l-m}} \\ & + \frac{cl!}{(m+1)!}\varepsilon^{[i}{}_{\alpha_1-}e_{\alpha_2-\dots\alpha_l-ik_1\dots k_m}{}^{r_1\dots r_{N-l-m}}A^{\alpha_1-\dots k_1\dots k_m]}{}_{r_1\dots r_{N-l-m}} \\ & + \frac{(a'+(N-l-m)!b')m!}{(l+1)!}2(\bar{\varepsilon}_{+k_1}\not{p})^{(\gamma-}e_{\gamma-\alpha_1-\dots\alpha_l-}{}^{r_1\dots r_{N-l-m}}_{k_2\dots k_m}A^{\alpha_1-\dots\alpha_l-)k_1\dots k_m}{}_{r_1\dots r_{N-l-m}} \\ & + \frac{c'l!}{(N-l-m+1)!}2(\bar{\varepsilon}_{+[i}\not{p}\eta)_{\alpha_1-}e_{\alpha_2-\dots\alpha_l-}{}^{ir_1\dots r_{N-l-m}}_{k_1\dots k_m}A^{\alpha_1-\dots\alpha_l-k_1\dots k_m}{}_{r_1\dots r_{N-l-m}]}. \end{aligned} \tag{4.3}$$

Taking terms in $e_{\alpha_1-\dots\alpha_l-}{}^{r_1\dots r_{N-l-m}}_{k_1\dots k_m}$ gives us on substitution and with a factor $(-1)^l$ from pulling the $\varepsilon$ through the basis function

$$\begin{aligned} \delta A^{\alpha_1-\dots\alpha_l-k_1\dots k_m}{}_{r_1\dots r_{N-l-m}} = {} & (-1)^{N+1}\frac{(N-l-m+1)(l+1)}{(1+l+m)l\cdot l!}\bar{\varepsilon}^{i(\alpha_1-}A^{\alpha_2-\dots\alpha_l-)k_1\dots k_m}{}_{ir_1\dots r_{N-l-m}} \\ & - \frac{l+1}{m!(l+2)}\varepsilon^{[k_1}_{\rho-}A^{\rho-\alpha_1-\dots k_2\dots k_m]\delta\text{-traceless}}{}_{r_1\dots r_{N-l-m}} \end{aligned}$$

$$+\frac{m+1}{l!\,l}(l+1)2(\bar{\varepsilon}_{+i}\not{p})^{(\alpha_1-}A^{\alpha_2-\ldots)ik_1\ldots k_m}{}_{r_1\ldots r_{N-l-m}}$$

$$+(-1)^{m+1}\frac{(l+1)(2+l+m)}{(l+2)(N-l-m)!}2(\bar{\varepsilon}_{+[r_1}\not{p}\eta)_{\rho-}A^{\rho-\alpha_1-\ldots\alpha_l-k_1\ldots k_m}{}_{r_2\ldots r_{N-l-m}]\delta\text{-traceless}}. \tag{4.4}$$

Here, we have had to apply trace conditions on two of the terms in order to keep the SU($N$) classification for the components. This involves adding extra terms which vanish on contraction with the basis function. For example,

$$\varepsilon^{[k_1}_{\rho-}A^{\rho-\alpha_1-\ldots\alpha_*-k_2\ldots k_m]\delta\text{-traceless}}{}_{r_1\ldots r_{N-l-m}}$$

$$=\varepsilon^{[k_1}_{\rho-}A^{\rho-\alpha_1-\ldots\alpha_l-k_2\ldots k_m]}{}_{r_1\ldots r_{N-l-m}}-\frac{1}{(l+2)}\sum_{s=1}^{N-l-m}\varepsilon^{j}_{\rho-}\delta^{[k_1}_{r_s}A^{\rho-\alpha_1-\ldots k_2\ldots]}{}_{r_1\ldots r_{s-1}jr_{s+1}\ldots r_{N-l-m}}.$$

Redefining

$$A^{\alpha_1-\ldots\alpha_l-k_1\ldots k_m}{}_{r_1\ldots r_{N-l-m}}=\prod_{s=1}^{l/2}(i\eta\gamma^{\mu'_s}\not{p})^{(\alpha_{2s-1}-\alpha_{2s}-)}A_{\mu'_1\ldots\mu'_{l/2}}{}^{\boldsymbol{k}}_{\boldsymbol{r}} \qquad l \text{ even}$$

$$=\prod_{s=1}^{l-1/2}(i\eta\gamma^{\mu'_s}\not{p})^{(\alpha_{2s-1}-\alpha_{2s}-}A^{\alpha_l-)}_{\mu'_1\ldots\mu'_{((l-1)/2)}}{}^{\boldsymbol{k}}_{\boldsymbol{r}} \qquad l \text{ odd} \tag{4.5}$$

with

$$p^{\mu_1}A_{\mu'_1\ldots}{}^{\boldsymbol{k}}_{\boldsymbol{r}}=p^{\mu_1}A^{\alpha-}_{\mu'_1\ldots}{}^{\boldsymbol{k}}_{\boldsymbol{r}}=(\gamma^{\mu_1}A_{\mu_1\ldots}{}^{\boldsymbol{k}}_{\boldsymbol{r}})_{\alpha-}=0=g^{\mu_1\mu_2}A_{\mu_1\mu_2\ldots}=g^{\mu_1\mu_2}A^{\alpha-}_{\mu_1\mu_2\ldots}$$

and using

$$\bar{\varepsilon}^{i(\alpha_1-}A^{\alpha_2-)\boldsymbol{k}}_{\boldsymbol{\mu}'\,i\boldsymbol{r}}=\frac{1}{2p^2}(i\eta\gamma^{\nu}\not{p})^{\alpha_1-\alpha_2-}(\bar{\varepsilon}^{i-}i\gamma_{\nu'}\not{p}A_{\boldsymbol{\mu}-}{}^{\boldsymbol{k}}_{i\boldsymbol{r}})$$

we get

$$\delta A_{\boldsymbol{\mu}'}{}^{k_1\ldots k_m}_{r_1\ldots r_{N-l-m}}=\frac{(-1)^{m+1}(N-l-m+1)(l+1)}{2l^2(1+l+m)(l/2)!}\frac{i}{p^2}(\bar{\varepsilon}^{i}\gamma_{(\mu'_1}\not{p}A_{\mu'_2\ldots)}{}^{\boldsymbol{k}}_{i\boldsymbol{r}})$$

$$+\frac{(l+2)^2}{m!(l+2)}(\bar{\varepsilon}^{[k_1-}A_{\boldsymbol{\mu}'}{}^{k_2\ldots k_m]}_{\boldsymbol{r}})^{\delta\text{-traceless}}-\frac{(m+1)(l+1)}{l^2(l/2)!}i(\bar{\varepsilon}_i\gamma_{(\mu'_1}A_{\mu'_2\ldots)}{}^{i\boldsymbol{k}}_{\boldsymbol{r}})$$

$$+\frac{(-1)^m(l+1)^2(2+l+m)}{(l+2)(N-l-m)!}2(\bar{\varepsilon}_{[r_1}\not{p}A_{\mu'_1\ldots}{}^{\boldsymbol{k}}_{r_2\ldots]})_{\delta\text{-traceless}} \tag{4.6a}$$

$$\delta A_{\boldsymbol{\mu}'\alpha-}{}^{k_1\ldots k_m}_{r_1\ldots r_{N-l-m}}$$

$$=\frac{(-1)^{m+1}(N-l-m+1)(l+1)}{(1+l+m)l^2}\varepsilon^{i}_{\alpha-}A_{\boldsymbol{\mu}'}{}^{\boldsymbol{k}}_{i\boldsymbol{r}}$$

$$+\frac{(l+1)^2}{m!(l+2)}i(\not{p}\gamma^{\nu'}\varepsilon^{[k_1}_{-})_{\alpha-}A_{\nu'\boldsymbol{\mu}}{}^{k_2\ldots k_m]\delta\text{-traceless}}_{\boldsymbol{r}}$$

$$-\frac{(m+1)(l+1)}{(l+2)}2(\not{p}\varepsilon_{+i})_{\alpha-}A_{\boldsymbol{\mu}'}{}^{i\boldsymbol{k}}_{\boldsymbol{r}}$$

$$+\frac{(-1)^m(l+1)^2(2+l+m)}{(l+2)(N-l-m)!}2i(\gamma^{\nu'}\varepsilon_{[r_1+})_{\alpha-}p^2A_{\nu'\boldsymbol{\mu}'}{}^{\boldsymbol{k}}_{r_2\ldots r_{N-l-m}]\delta\text{-traceless}}. \tag{4.6b}$$

However, we want (4.6*b*) to be $\gamma$-traceless as well so we need to modify this slightly (again the modification vanishes when contracted with the respective basis function). The terms affected are the first and third, e.g.

$$\varepsilon^{\,i}_{\alpha -}A_{\mu' ir}{}^{k} \to \varepsilon^{\,i}_{\alpha -}A_{\mu' ir}{}^{k} - \frac{1}{l}\sum_{s=1}^{(l-1)/2}(\gamma_{\mu_s}\gamma^{\nu'}\varepsilon^{\,i}_{-})_{\alpha -}A_{\mu_1\ldots\mu_{s-1}\nu\mu_{s+1}\ldots ir}{}^{k} \qquad (4.6c)$$

giving $(\gamma \cdot \delta A) = 0$ as required.

Finally, we redefine

$$B_{\mu' r_{N-l-m}}{}^{k_m} = (p^2)^{[(l+m+1)/2]}A_{\mu' r}{}^{k}\left(\frac{(l!)^2}{(l+1)!}(1+l+m)!\right)2^{l+m}$$

$$\psi_{\mu'\alpha r_{N-l-m}}{}^{k_m} = ((\not p)^{l+m}A_{\mu' r}{}^{k})_\alpha\left(\frac{(l!)^2}{(l+1)!}(1+l+m)!\right)2^{l+m} \qquad (4.7)$$

$$= \not p_\alpha^{\beta_1}\ldots \not p_{\beta_{l+m-1}}{}^{\beta_{l+m}}A_{\mu'\beta_{l+m}-r}{}^{k}\left(\frac{(l!)^2}{(l+1)!}(1+l+m)!\right)2^{l+m}$$

giving $\pm$ chirality for $m$ even/odd.

This gives us, for even $m$,

$$\delta B_{\mu r}{}^{k} = \frac{(-1)^{m+1}(N-l-m+1)}{(l/2)!}i\bar\varepsilon^{\,i}_{-}\gamma_{(\mu_1}\psi_{+\mu_2\ldots)ir}{}^{k}$$
$$+\frac{1}{m!}\left(\bar\varepsilon_{-}^{[k_1}\psi_{-\mu' r}{}^{k_2\ldots]} - \frac{1}{l+2}\bar\varepsilon^{\,j}_{-}\sum_{s=1}^{N-m-l}\delta^{[k_1}_{r_s}\psi_{-\mu_1\ldots r_1\ldots r_{s-1}jr_s\ldots r_{N-l-m}}{}^{k_2\ldots k_m]}\right)$$
$$-\frac{m+1}{(l/2)!}i\bar\varepsilon_{+i}\gamma_{(\mu_1}\psi_{-\mu_2\ldots)r}{}^{ik} + \frac{(-1)^m}{(N-l-m)!}\left(\bar\varepsilon_{+[r_1}\psi_{+\mu_1\ldots r_2\ldots r_{N-l-m}]}{}^{k_1\ldots k_m}\right.$$
$$\left.-\frac{1}{l+2}\sum_{s=1}^{m}\bar\varepsilon_{+j}\delta^{k_1}_{r_1}\psi_{+\mu_1\ldots r_2\ldots r_{N-l-m}]}{}^{k_2\ldots k_{s-1}jk_{s+1}\ldots k_m}\right) \qquad (4.8a)$$

$$\delta\psi_{\mu\alpha+r_{N-l-m}}{}^{k_m} = (-1)^{m+1}2(N-l-m+1)\Bigg((\not p\varepsilon^{\,i}_{-})_{\alpha+}B_{\mu' ir}{}^{k}$$
$$\left.-\frac{1}{l}\sum_{s=1}^{(l-1)/2}(\gamma_{\mu_s}\gamma^{\nu}\not p\varepsilon^{\,i}_{-})_{\alpha+}B_{\mu_1\ldots\mu_{s-1}\nu\mu_{s+1}\ldots ir}{}^{k}\right)$$
$$+\frac{1}{m!}i\Bigg((\gamma^{\nu'}\varepsilon_{-}^{[k_1})_{\alpha+}B_{\nu'\mu' r}{}^{k_2\ldots k_m]}$$
$$\left.-\frac{1}{l+2}\sum_{s=1}^{N-l-m}(\gamma^{\nu}\varepsilon^{\,j}_{-})_{\alpha+}\delta^{[k_1}_{r_1}B_{\nu'\mu r_2\ldots r_{s-1}jr_{s+1}\ldots r_{N-l-m}}{}^{k_2\ldots k_m]}\right)$$
$$\left.-2(m+1)\;\varepsilon_{\alpha+i}B_{\mu' r}{}^{ik} - \frac{1}{l}\sum_{s=1}^{(l-1)/2}(\gamma_{\mu_s}\gamma^{\nu}\varepsilon_{+i})_{\alpha+}B_{\mu_1\ldots\mu_{s-1}\nu\mu_{s+1}\ldots r}{}^{ik}\right)$$
$$+\frac{(-1)^m}{(N-l-m)!}i\Bigg((\not p\gamma^{\nu}\varepsilon_{+[r_1})_{\alpha+}B_{\nu'\mu' r_2\ldots]}{}^{k_1\ldots k_m}$$
$$\left.-\frac{1}{l+2}\sum_{s=1}^{m}(\not p\gamma^{\nu}\varepsilon_{-j})_{\alpha+}\delta^{k_s}_{[r_1}B_{\nu'\mu' r_2\ldots]}{}^{k_1\ldots k_{s-1}jk_{s+1}\ldots k_m}\right) \qquad (4.8b)$$

plus similar terms for $m$ odd, each with extra factors of $\not p$ in appropriate places.

However, for even $N$ we have the double $\varepsilon$ condition on the $m = N/2$ basis function. This carries over onto the field component and we get the transformation rules

$$\delta B^{k_1 \ldots k_{N/2}}_{r \ldots r_{N/2}} = \frac{1}{2}\frac{1}{(N/2)!}(\bar{\varepsilon}_-^{[k_1}\psi_-{}^{k_2 \ldots k_{N/2}]}_{r_1 \ldots r_{N/2}} + \bar{\varepsilon}_{+[r_1}\psi_+{}^{k_1 \ldots k_{N/2}}_{r_2 \ldots r_{N/2}]})$$
$$+\frac{1}{2}\frac{(-1)^{N/2}}{[(N/2)!]^2}\varepsilon^{k_1 \ldots k_{N/2}\, l_1 \ldots l_{N/2}}\varepsilon_{r_1 \ldots r_{N/2}\, s_1 \ldots s_{N/2}}(\bar{\varepsilon}_-^{s_1}\psi_-{}^{s_2 \ldots s_{N/2}}_{r_1 \ldots r_{N/2}} + \bar{\varepsilon}_{+r_1}\psi_+{}^{k_1 \ldots k_{N/2}}_{r_2 \ldots r_{N/2}})$$

with the extra factor $\not{p}$ should $N/2$ be odd.

Taking $N$ even, we get the reality condition mentioned in § 3:

$$F\phi = (16p^2)^{N/2}\phi^* = (16p^2)^{N/2}(e_-(A_+)^* + e_+(A_-)^*) = (Fe_-)A_+ + (Fe_+)A_- \tag{4.9}$$

giving

$$(16p^2)^{N/2}(A^{\alpha_1 - \ldots \alpha_2 -}{}^{r_1 \ldots r_{N-l-m}}_{-k_1 \ldots k_m})^* = C(N, l, m)2^l \prod_{s=1}^{l} \not{p}_{\beta_s -}{}^{\alpha_s +} A^{\beta_1 - \ldots \beta_l -}{}^{k_1 \ldots k_m}_{r_1 \ldots r_{N-l-m}}.$$

For the bosons we get

$$(B_{\mu}{}^{k_1 \ldots k_m}_{r_1 \ldots r_{N-l-m}})^* = (-1)^m B_{\mu}{}^{r_1 \ldots r_{N-l-m}}_{k_1 \ldots k_m} \tag{4.10}$$

and for fermions

$$(\psi_{\mu\alpha\pm}{}^k_r)^* = \psi_{\mu\alpha\mp}{}^r_k. \tag{4.11}$$

Hence we get important reality conditions on the bosons when $m = N - l - m$, e.g. $(B_{\mu'\nu'})^* = B_{\mu'\nu'}$ in the $N = 4$ case.

Before we give the invariant Lagrangian for the transformation rules, we need to consider the dimensions of the field components. From (4.8*a*, *b*) it is apparent that the fermions share the same dimensionality, $d$, and that the bosons have dimensions $d - \frac{1}{2}$, $d + \frac{1}{2}$ for $m$ odd/even respectively. Setting $d = -\frac{3}{2}$ we get physical bosons for $m$ even and auxiliary for $m$ odd and the Lagrangian looks like

$$\mathscr{L} = \sum_{l,m} f(l, m)\overline{(\psi_{\mu'}{}^r_k)^*}\not{p}\psi_{\mu r_{N-l-m}}{}^{k_m} + b(l, m = 2n)(B_{\mu' r_{N-l-m}}{}^{k_m})^* p^2 B_{\mu' r}{}^k$$
$$+ b(l, m = 2n+1)(B_{\mu' r_{N-l-m}}{}^{k_m})^* B_{\mu' r}{}^k. \tag{4.12}$$

The rules (4.8*a*, *b*) then give us values for the coefficients

$$f(l, m) = -(-\tfrac{1}{2})^{(l+1)/2} m!(N - l - m)! \qquad b(l, m) = (-\tfrac{1}{2})^{l/2} m!(N - l - m)! \tag{4.13}$$

where an extra factor of $\frac{1}{2}$ is needed in $b(l, m)$ when $2m + l = N$ and $N$ is even, so that the boson is real in the real irrep. We have now given the transformation laws and Lagrangian for the general $N$, $Y = 0$ multiplet. It is possible in some cases to simplify the coefficients further but we feel that this detracts from the general nature of the equations.

## 5. Examples

(*a*) $N=3$: $Y=0$

We get the field components

$$\psi_{\mu\alpha+},\ \psi_{r_1r_2\alpha+},\ \psi^{k}_{r\alpha-},\ \psi^{k_1k_2}_{\alpha+}$$

$$B_{\mu r},\ B^{k}_{\mu},\ B_{r_1r_2r_3},\ B^{k}_{r_1r_2},\ B^{k_1k_2}_{r},\ B^{k_1k_2k_3}$$

with the following transformation rules:

$$\delta B_{r_1r_2r_3}=(1/3!)\bar\varepsilon_{+[r_1}\psi_{+r_2r_3]}$$

$$\delta B^{k}_{r_1r_2}=(1/1!)(\bar\varepsilon^{k}_{-}\not p\psi_{+r_1r_2}-\tfrac12\bar\varepsilon^{j}_{-}\not p(\delta^{k}_{r_1}\psi_{+jr_2}+\delta^{k}_{r_2}\psi_{r_1j}))$$
$$-(1/2!)(\bar\varepsilon_{+[r_1}\not p\psi^{k}_{-r_2]}-\tfrac12\bar\varepsilon_{+j}\not p\delta^{k}_{[r_1}\psi^{j}_{-r_2]})$$

$$\delta B^{k_1k_2}_{r}=(1/2!)(\bar\varepsilon^{[k_1}_{-}\psi^{k_2]}_{-r}-\tfrac12\bar\varepsilon^{j}_{-}\delta^{[k_1}_{r}\psi^{k_2]}_{-j})+(1/1!)(\bar\varepsilon_{+r}\psi^{k_1k_2}_{+}-\tfrac12\bar\varepsilon_{+j}(\delta^{k_1}_{r}\psi^{jk_2}_{+}+\delta^{k_2}_{r}\psi^{k_1j}_{+}))$$

$$\delta B^{k_1k_2k_3}=(1/3!)\bar\varepsilon^{[k_1}_{-}\not p\psi^{k_2k_3]}_{+}$$

$$\delta B_{\mu r}=-2i\bar\varepsilon^{i}_{-}\gamma_{\mu'}\psi_{+ir}-i\bar\varepsilon_{+i}\gamma_{\mu'}\psi^{i}_{-r}+(1/1!)\bar\varepsilon_{+r}\psi_{+\mu}$$

$$\delta B^{k}_{\mu}=i\bar\varepsilon^{i}_{-}\gamma_{\mu'}\not p\psi^{k}_{-i}+(1/1!)\bar\varepsilon^{k}_{-}\not p\psi_{+\mu}-2i\bar\varepsilon_{+i}\gamma_{\mu'}\not p\psi^{ik}_{+}$$

$$\delta\psi_{r_1r_2\alpha+}=-6(\not p\varepsilon^{i}_{-})_{\alpha+}B_{ir_1r_2}-2\varepsilon_{\alpha+i}B^{i}_{r_1r_2}+(i/2!)(\not p\gamma^{\nu}\varepsilon_{+[r_1})_{\alpha+}B_{\nu r_2]}$$

$$\delta\psi^{k}_{r\alpha-}=4\varepsilon^{i}_{\alpha-}B^{k}_{ir}+(1/1!)i((\not p\gamma^{\nu}\varepsilon^{k}_{-})_{\alpha-}B_{\nu r}-\tfrac13(\not p\gamma^{\nu}\varepsilon^{j}_{-})_{\alpha-}\delta^{k}_{r}B_{\nu j})$$
$$-4(\not p\varepsilon_{+i})_{\alpha-}B^{ik}_{r}-(1/1!)i((\gamma^{\nu}\varepsilon_{+r})_{\alpha-}B^{k}_{\nu}-\tfrac13(\gamma^{\nu}\varepsilon_{+j})_{\alpha-}\delta^{k}_{r}B^{j}_{\nu})$$

$$\delta\psi^{k_1k_2}_{\alpha+}=-2(\not p\varepsilon^{i}_{-})_{\alpha+}B^{k_1k_2}_{i}+(1/2!)i(\gamma^{\nu}\varepsilon^{[k_1}_{-})_{\alpha+}B^{k_2]}_{\nu}-6\varepsilon_{\alpha+i}B^{ik_1k_2}$$

$$\delta\psi_{\mu\alpha+}=-2(\not p\varepsilon^{i}_{-})_{\alpha+}B_{\mu i}-\tfrac13(\gamma_{\mu'}\gamma^{\nu}\not p\varepsilon^{i}_{-})_{\alpha+}B_{\nu i})-2(\varepsilon_{\alpha+i}B^{i}_{\mu}-\tfrac13(\gamma_{\mu'}\gamma^{\nu}\varepsilon_{+i})_{\alpha+}B^{i}_{\nu})$$

$$\mathscr{L}=3!(B_{r_1r_2r_3})^{*}p^{2}B_{r_1r_2r_3}+2!(B^{k}_{r_1r_2})^{*}B^{k}_{r_1r_2}+2!(B^{k_1k_2}_{r})^{*}p^{2}B^{k_1k_2}_{r}+3!(B^{k_1k_2k_3})^{*}B^{k_1k_2k_3}$$
$$-\tfrac12(B_{\mu r})^{*}p^{2}B_{\mu r}-\tfrac12(B^{k}_{\mu})^{*}B^{k}_{\mu}+\tfrac12 2!\overline{(\psi_{+r_1r_2})^{*}}\not p\psi_{+r_1r_2}+\tfrac12\overline{(\psi^{k}_{r-})^{*}}\not p\psi^{k}_{r-}$$
$$+\tfrac12 2!\overline{(\psi^{k_1k_2}_{+})^{*}}\not p\psi^{k_1k_2}_{+}-\tfrac14\overline{(\psi_{\mu+})^{*}}\not p\psi_{\mu+}.$$

(*b*) $N=4$: $Y=0$

We consider the $Y=0$ representation with the component field of spin 2 to have dimension $-2$. We therefore need the bosons with $m$ even to be auxiliary. This we do by taking $d=-\frac52$ and redefining the other fields:

$$\psi\to\not p\psi\qquad\qquad B(m\ \text{odd})\to p^{2}B.$$

Since we have a reality condition on the components (4.10), (4.11) our transformation rules are:

$$\delta B_{r_1r_2r_3r_4}=(1/4!)\bar\varepsilon_{+[r_1}\not p\psi_{-r_2r_3r_4]}=\delta(B^{r_1r_2r_3r_4})^{*}$$

$$\delta B^{k}_{r_1r_2r_3}=(\bar\varepsilon^{k}_{-}\psi_{-r_1r_2r_3}-\tfrac12\bar\varepsilon^{j}_{-}(\delta^{k}_{r_1}\psi_{-jr_2r_3}+\delta^{k}_{r_2}\psi_{-r_1jr_3}+\delta^{k}_{r_3}\psi_{-r_1r_2j}))$$
$$-(1/3!)(\bar\varepsilon_{+[r_1}\psi^{k}_{+r_2r_3]}-\tfrac12\bar\varepsilon_{+j}\delta^{k}_{[r_1}\psi^{j}_{+r_2r_3]})=-\delta(B^{r_1r_2r_3}_{n})^{*}$$

$$\delta B^{k_1k_2}_{r_1r_2}=\tfrac14(\bar\varepsilon^{[k_1}_{-}\not p\psi^{k_2]}_{+r_1r_2}+\bar\varepsilon_{+[r_1}\not p\psi^{k_1k_2}_{-r_2]})$$
$$+\tfrac14\varepsilon^{k_1k_2k_3k_4}\varepsilon_{r_1r_2r_3r_4}(\bar\varepsilon^{k_3}_{-}\not p\psi^{k_4}_{r_2r_4}+\bar\varepsilon_{+r_3}\not p\psi^{k_3k_4}_{-r_4})=\delta(B^{r_1r_2}_{k_1k_2})^{*}$$

$$\delta B_{\mu r_1 r_2} = -3i\bar{\varepsilon}^{\,i}_{-}\gamma_{\mu'}\not{p}\psi_{-ir_1r_2} - i\bar{\varepsilon}_{+i}\gamma_{\mu'}\not{p}\psi^{\,i}_{+r_1r_2} + \tfrac{1}{2}\bar{\varepsilon}_{+[r_1}\not{p}\psi_{-\mu r_2]} = \delta(B_{\mu}^{\,r_1r_2})^*$$

$$\begin{aligned}\delta B_{\mu r}^{\ \ k} &= 2i\bar{\varepsilon}^{\,i}_{-}\gamma_{\mu'}\psi^{\,k}_{+ir} + (\bar{\varepsilon}^{\,k}_{-}\psi_{-\mu r} - \tfrac{1}{4}\bar{\varepsilon}^{\,j}_{-}\delta^{k}_{r}\psi_{-\mu j})\\ &\quad - 2i\bar{\varepsilon}_{+i}\gamma_{\mu'}\psi^{\,ik}_{-\mu} - (\bar{\varepsilon}_{+r}\psi^{\,k}_{+\mu} - \tfrac{1}{4}\bar{\varepsilon}_{+j}\delta^{k}_{r}\psi^{\,j}_{+\mu}) = -\delta(B_{\mu k}^{\ \ r})^*\end{aligned}$$

$$\delta B_{\mu\nu} = -\tfrac{1}{2}i\bar{\varepsilon}^{\,i}_{-}\gamma_{(\mu'}\not{p}\psi_{\nu)-i} - \tfrac{1}{2}i\bar{\varepsilon}_{+i}\gamma_{(\mu'}\not{p}\psi^{\,i}_{\nu)+} = \delta(B_{\mu\nu})^*$$

$$\begin{aligned}\delta\psi_{\nu'\alpha-r} &= -4(\varepsilon^{\,i}_{\alpha-}B_{\nu' ir} - \tfrac{1}{3}(\gamma_{\nu'}\gamma^{\mu}\varepsilon^{\,i}_{-})_{\alpha-}B_{\mu ir}) + i(\gamma^{\nu}\varepsilon_{-r})_{\alpha-}B_{\nu\mu}\\ &\quad - 2((\not{p}\varepsilon_{+i})_{\alpha-}B_{\nu'}{}^{i}_{r} - \tfrac{1}{3}(\gamma_{\nu'}\gamma^{\mu}\not{p}\varepsilon_{+i})_{\alpha-}B_{\mu'}{}^{i}_{r})\end{aligned}$$

$$\delta\psi_{\alpha-r_1r_2r_3} = -8\varepsilon^{\,i}_{\alpha-}B_{ir_1r_2r_3} - 2(\not{p}\varepsilon_{+i})_{\alpha-}B^{\,i}_{r_1r_2r_3} + \tfrac{1}{6}i(\gamma^{\nu}\varepsilon_{+[r_1})_{\alpha-}B_{\nu r_2r_3]}$$

$$\begin{aligned}\delta\psi_{\alpha+r_1r_2}^{\ \ k} &= 6(\not{p}\varepsilon^{\,i}_{-})_{\alpha+}B^{\,k}_{ir_1r_2} + ((\gamma^{\nu}\varepsilon^{\,k}_{-})_{\alpha+}B_{\nu r_1r_2} - \tfrac{1}{3}(\gamma^{\nu}\varepsilon^{\,j}_{-})_{\alpha+}(\delta^{k}_{r_1}B_{\nu jr_2} + \delta^{k}_{r_2}B_{\nu r_1 j}))\\ &\quad - 4\varepsilon_{\alpha+i}B^{\,ik}_{r_1r_2} - \tfrac{1}{2}((\not{p}\gamma^{\nu}\varepsilon_{+[r_1})_{\alpha+}B^{\,k}_{\nu r_2]} - \tfrac{1}{3}(\not{p}\gamma^{\nu}\varepsilon_{+j})_{\alpha+}\delta^{k}_{[r_1}B^{\,j}_{\nu r_2]})\end{aligned}$$

when we take account of the reality conditions

$$(B^{\,k_1k_2}_{r_1r_2})^* = B^{\,r_1r_2}_{k_1k_2} \qquad (B_{\mu\nu})^* = B_{\mu\nu} \qquad (B^{\,k}_{\mu\nu})^* = -B_{\mu k}^{\ \ r}.$$

Our Lagrangian is then

$$\begin{aligned}\mathscr{L} &= 4!B^{r_1r_2r_3r_4}B_{r_1r_2r_3r_4} - 3!B_{k}^{\,r_1r_2r_3}p^2B^{\,k}_{r_1r_2r_3} + \tfrac{1}{2}2!2!B^{\,r_1r_2}_{k_1k_2}B^{\,k_1k_2}_{r_1r_2} - \tfrac{1}{2}2!B_{\mu}^{\,r_1r_2}B_{\mu r_1r_2}\\ &\quad + \tfrac{1}{4}B_{\mu k}^{\ \ r}p^2B_{\mu r}^{\ \ k} + \tfrac{1}{8}B_{\mu\nu}B_{\mu\nu} + \tfrac{1}{2}3!\bar{\psi}_{+}^{\,r_1r_2r_3}\not{p}\psi_{-r_1r_2r_3}\\ &\quad + \tfrac{1}{2}2!\bar{\psi}^{\,r_1r_2}_{-k}\not{p}\psi^{\,k}_{+r_1r_2} - \tfrac{1}{4}\bar{\psi}^{\,r}_{\mu+}\not{p}\psi_{\mu-r}.\end{aligned}$$

(*c*) $N = 6$: $Y = 0$

We consider here only the irrep with maximum spin 3 having dimension $-1$. Since this is a straightforward application of our formulae, we just give the table of the coefficients in the order in which they occur in equations (4.8):

| Field | $l$ | $m$ | $I$ | $II$ | $III$ | $IV$ |
|---|---|---|---|---|---|---|
| $B_{r_1\ldots r_6}$ | 0 | 0 | 0 | 0 | 0 | 1/6! |
| $B^{k}_{r_1\ldots r_5}$ | 0 | 1 | 0 | 1/1! | 0 | −1/5! |
| $B^{k_1k_2}_{r_1\ldots r_4}$ | 0 | 2 | 0 | 1/2! | 0 | 1/4! |
| $B^{k_1k_2k_3}_{r_1r_2r_3}$ | 0 | 3 | 0 | 1/3!2 | 0 | −1/3!2 |
| $B_{\mu r_1\ldots r_4}$ | 2 | 0 | −5 | 0 | −1 | 1/4! |
| $B_{\mu r_1r_2r_3}^{\ \ k}$ | 2 | 1 | 4 | 1/1! | −2 | −1/3! |
| $B_{\mu r_1r_2}^{\ \ k_1k_2}$ | 2 | 2 | −3 | 1/2! | −3 | 1/2! |
| $B_{\mu_1\mu_2r_1r_2}$ | 4 | 0 | −3/2! | 0 | −1/2! | 1/2! |
| $B_{\mu_1\mu_2r}^{\ \ k}$ | 4 | 1 | 2/2! | 1/1! | −2/2! | −1/1! |
| $B_{\mu_1\mu_2\mu_3}$ | 6 | 0 | −1/3! | 0 | −1/3! | 0 |
| $\psi_{\alpha+r_1\ldots r_5}$ | 1 | 0 | −12 | 0 | −2 | 1/5! |
| $\psi_{\alpha-r_1\ldots r_4}^{\ \ k}$ | 1 | 1 | 10 | 1/1! | −4 | −1/4! |
| $\psi_{\alpha-r_1r_2r_3}^{\ \ k_1k_2}$ | 1 | 2 | −8 | 1/2! | −6 | 1/3! |
| $\psi_{\mu\alpha-r_1r_2r_3}$ | 3 | 0 | −8 | 0 | −2 | 1/3! |
| $\psi_{\mu\alpha-r_1r_2}^{\ \ k}$ | 3 | 1 | 6 | 1 | −4 | −1/2! |
| $\psi_{\mu_1\mu_2\alpha-r}$ | 5 | 0 | −4 | 0 | −2 | 1 |

with all other fields given by the reality conditions (4.10) and (4.11).

Our Lagrangian is

$$
\begin{aligned}
\mathscr{L} = {} & 6!B^{r_1\cdots r_6}p^2B_{r_1\ldots r_6} - 5!B_k^{r_1\cdots r_5}B^k_{r_1\ldots r_5} \\
& + 2!4!B^{r_1\ldots r_4}_{k_1k_2}p^2B^{k_1k_2}_{r_1\ldots r_4} - \tfrac{1}{2}3!3!B^{r_1r_2r_3}_{k_1k_2k_3}B^{k_1k_2k_3}_{r_1r_2r_3} \\
& - \tfrac{1}{2}4!B_\mu^{r_1\cdots r_4}p^2B_{\mu r_1\ldots r_4} + \tfrac{1}{2}3!B_{\mu k}^{\ \ r_1r_2r_3}B_{\mu r_1r_2r_3}^{\ \ \ \ \ \ \ \ k} \\
& - \tfrac{1}{4}2!2!B_{\mu\ k_1k_2}^{\ r_1r_2}p^2B_{\mu\ r_1r_2}^{\ k_1k_2} + \tfrac{1}{4}2!B^{r_1r_2}_{\mu_1\mu_2}p^2B_{\mu_1\mu_2r_1r_2} \\
& - \tfrac{1}{8}B_{\mu_1\mu_2k}^{\ \ \ \ \ \ r}B_{\mu_1\mu_2r}^{\ \ \ \ \ \ k} - \tfrac{1}{16}B_{\mu_1\mu_2\mu_3}p^2B_{\mu_1\mu_2\mu_3} + \tfrac{1}{2}5!\bar{\psi}_-^{r_1\cdots r_5}\not{p}\psi_{+r_1\ldots r_5} \\
& + \tfrac{1}{2}4!\bar{\psi}_{+k}^{\ \ r_1\ldots r_4}\not{p}\psi^{\ \ k}_{-r_1\ldots r_4} + \tfrac{1}{2}2!3!\bar{\psi}_{-k_1k_2}^{\ \ r_1r_2r_3}\not{p}\psi^{\ \ k_1k_2}_{+r_1r_2r_3} - \tfrac{1}{4}3!\bar{\psi}_{\mu-}^{\ \ r_1r_2r_3}\not{p}\psi_{\mu+r_1r_2r_3} \\
& - \tfrac{1}{4}2!\bar{\psi}_{\mu+k}^{\ \ \ \ \ r_1r_2}\not{p}\psi_{\mu-r_1r_2}^{\ \ \ \ \ \ \ k} + \tfrac{1}{8}\bar{\psi}^{\ r}_{\mu_1\mu_2-}\not{p}\psi_{\mu_1\mu_2+r}.
\end{aligned}
$$

(*d*) $N=8$: $Y=0$

We again just give the table of coefficients and the Lagrangian for the $Y=0$ irrep with maximum spin having dimension $-1$. We also have a reality condition on the fields given by (4.10) and (4.11).

| Field | $l$ | $m$ | $I$ | $II$ | $III$ | $IV$ |
|---|---|---|---|---|---|---|
| $B_{r_1\ldots r_8}$ | 0 | 0 | 0 | 0 | 0 | 1/8! |
| $B^k_{r_1\ldots r_7}$ | 0 | 1 | 0 | 1/1! | 0 | −1/7! |
| $B^{k_1k_2}_{r_1\ldots r_6}$ | 0 | 2 | 0 | 1/2! | 0 | 1/6! |
| $B^{k_1k_2k_3}_{r_1\ldots r_5}$ | 0 | 3 | 0 | 1/3! | 0 | −1/5! |
| $B^{k_1\cdot k_4}_{r_1\ldots r_4}$ | 0 | 4 | 0 | 1/4!2 | 0 | 1/4!2 |
| $B_{\mu r_1\ldots r_6}$ | 2 | 0 | −7 | 0 | −1 | 1/6! |
| $B_{\mu r_1\ldots r_5}^{\ \ k}$ | 2 | 1 | 6 | 1/1! | −2 | −1/5! |
| $B_{\mu\ r_1\ldots r_4}^{\ k_1k_2}$ | 2 | 2 | −5 | 1/2! | −3 | 1/4! |
| $B_{\mu\ r_1r_2r_3}^{\ k_1k_2k_3}$ | 2 | 3 | 4 | 1/3! | −4 | −1/3! |
| $B_{\mu_1\mu_2r_1\ldots r_4}$ | 4 | 0 | −5/2! | 0 | −1/2! | 1/4! |
| $B_{\mu_1\mu_2r_1r_2r_3}^{\ \ \ \ \ \ k}$ | 4 | 1 | 4/2! | 1/1! | −2/2! | −1/3! |
| $B_{\mu_1\mu_2r_1r_2}^{\ \ \ \ \ \ k_1k_2}$ | 4 | 2 | −3/2! | 1/2! | −3/2! | 1/2! |
| $B_{\mu_1\mu_2\mu_3r_1r_2}$ | 6 | 0 | −3/3! | 0 | −1/3! | 1/2! |
| $B_{\mu_1\mu_2\mu_3r}^{\ \ \ \ \ \ \ \ k}$ | 6 | 1 | 2/3! | 1/1! | −2/3! | −1/1! |
| $B_{\mu_1\mu_2\mu_3\mu_4}$ | 8 | 0 | −1/4! | 0 | −1/4! | 0 |
| $\psi_{\alpha+r_1\ldots r_7}$ | 1 | 0 | −16 | 0 | −2 | 1/7! |
| $\psi_{\alpha-r_1\ldots r_6}^{\ \ \ \ k}$ | 1 | 1 | 14 | 1/1! | −4 | −1/6! |
| $\psi_{\alpha+r_1\ldots r_5}^{\ \ \ \ k_1k_2}$ | 1 | 2 | −12 | 1/2! | −6 | 1/5! |
| $\psi_{\alpha-r_1\ldots r_4}^{\ \ \ \ k_1k_2k_3}$ | 1 | 3 | 10 | 1/3! | −8 | −1/4! |
| $\psi_{\mu\alpha+r_1\ldots r_5}$ | 3 | 0 | −12 | 0 | −2 | 1/5! |
| $\psi_{\mu\alpha-r_1\ldots r_4}^{\ \ \ \ \ \ k}$ | 3 | 1 | 10 | 1/1! | −4 | −1/4! |
| $\psi_{\mu\alpha+r_1r_2r_3}^{\ \ \ \ \ \ k_1k_2}$ | 3 | 2 | −8 | 1/2! | −6 | 1/3! |
| $\psi_{\mu_1\mu_2\alpha+r_1r_2r_3}$ | 5 | 0 | −8 | 0 | −2 | 1/3! |
| $\psi_{\mu_1\mu_2\alpha-r_1r_2}^{\ \ \ \ \ \ \ \ k}$ | 5 | 1 | 6 | 1 | −4 | −1/2! |
| $\psi_{\mu_1\mu_2\mu_3\alpha+r}$ | 7 | 0 | −4 | 0 | −2 | 1 |

$$
\begin{aligned}
\mathscr{L} = {} & 8!B^{r_1\cdots r_8}p^2B_{r_1\ldots r_8} - 7!B_k^{r_1\cdots r_7}B^k_{r_1\ldots r_7} \\
& + 2!6!B^{r_1\ldots r_6}_{k_1k_2}p^2B^{k_1k_2}_{r_1\ldots r_6} - 3!5!B^{r_1\cdots r_5}_{k_1k_2k_3}B^{k_1k_2k_3}_{r_1\ldots r_5} \\
& + \tfrac{1}{2}4!4!B^{r_1\ldots r_4}_{k_1\ldots k_4}p^2B^{k_1\ldots k_4}_{r_1\ldots r_4} - \tfrac{1}{2}6!B_\mu^{r_1\cdots r_6}p^2B_{\mu r_1\ldots r_6} + \tfrac{1}{2}5!B_{\mu k}^{\ \ r_1\cdots r_5}B_{\mu r_1\ldots r_5}^{\ \ \ \ \ \ \ \ k}
\end{aligned}
$$

$$
\begin{aligned}
&-\tfrac{1}{2}2!\,4!\,B_{\mu\, k_1k_2}^{\;\; r_1\ldots r_4}p^2B_{\mu\, r_1\ldots r_4}^{\;\; k_1k_2}+\tfrac{1}{4}3!\,3!\,B_{\mu\, k_1k_2k_3}^{\;\; r_1r_2r_3}B_{\mu\, r_1r_2r_3}^{\;\; k_1k_2k_3}\\
&+\tfrac{1}{4}4!\,B^{r_1\ldots r_4}_{\mu_1\mu_2}p^2B_{\mu_1\mu_2r_1\ldots r_4}-\tfrac{1}{4}3!\,B_{\mu_1\mu_2 k}^{\;\; r_1r_2r_3}B_{\mu_1\mu_2r_1r_2r_3}^{\;\; k}\\
&+\tfrac{1}{8}2!\,2!\,B_{\mu_1\mu_2k_1k_2}^{\;\; r_1r_2}p^2B_{\mu_1\mu_2r_1r_2}^{\;\; k_1k_2}-\tfrac{1}{8}2!\,B^{r_1r_2}_{\mu_1\mu_2\mu_3}p^2B_{\mu_1\mu_2\mu_3r_1r_2}\\
&+\tfrac{1}{16}B_{\mu_1\mu_2\mu_3k}^{\;\; r}B_{\mu_1\mu_2\mu_3r}^{\;\; k}+\tfrac{1}{32}B_{\mu_1\mu_2\mu_3\mu_4}p^2B_{\mu_1\mu_2\mu_3\mu_4}\\
&+\tfrac{1}{2}7!\,\bar\psi^{r_1\ldots r_7}_{-}\not p\psi_{+r_1\ldots r_7}+\tfrac{1}{2}6!\,\bar\psi^{r_1\ldots r_6}_{+k}\not p\psi^{k}_{-r_1\ldots r_6}\\
&+\tfrac{1}{2}2!\,5!\,\bar\psi^{r_1\ldots r_5}_{-k_1k_2}\not p\psi^{k_1k_2}_{+r_1\ldots r_5}+\tfrac{1}{2}3!\,4!\,\bar\psi^{r_1\ldots r_4}_{+k_1k_2k_3}\not p\psi^{k_1k_2k_3}_{-r_1r_2r_3r_4}\\
&-\tfrac{1}{4}5!\,\bar\psi^{r_1\ldots r_5}_{\mu-}\not p\psi_{\mu+r_1\ldots r_5}-\tfrac{1}{4}4!\,\bar\psi_{\mu+k}^{\;\; r_1\ldots r_4}\not p\psi_{\mu-r_1\ldots r_4}^{\;\; k}\\
&-\tfrac{1}{4}2!\,3!\,\bar\psi_{\mu-k_1k_2}^{\;\; r_1r_2r_3}\not p\psi_{\mu+r_1r_2r_3}^{\;\; k_1k_2}+\tfrac{1}{8}3!\,\bar\psi^{r_1r_2r_3}_{\mu_1\mu_2-}\not p\psi_{\mu_1\mu_2+r_1r_2r_3}\\
&+\tfrac{1}{8}2!\,\bar\psi_{\mu_1\mu_2+k}^{\;\; r_1r_2}\not p\psi_{\mu_1\mu_2-r_1r_2}^{\;\; k}-\tfrac{1}{16}\bar\psi^{r}_{\mu_1\mu_2\mu_3-}\not p\psi_{\mu_1\mu_2\mu_3+r}.
\end{aligned}
$$

## 6. Discussion

In order to use the irreps presented so far to build the off-shell associated supergravities, it is necessary to construct further irreps with higher values of $Y$ and SU($N$) content. In particular, for $N=4$ SYM it is necessary to take the SU(4) 15-dimensional adjoint representation attached to all the component fields in the $N=4$ irrep given in the last section. The resulting irrep will then have the correct SU(4) assignments for a singlet spin 1, a quartet of spin $\frac{1}{2}$ and a **6** of spin 0 (Taylor 1982b, c, Taylor and Rivelles 1982, Rivelles and Taylor 1982d). The remaining fields will have to be removed by appropriate central charge irreps which have to be constructed by different techniques from those used in the present paper. This is because, in the presence of central charges, chirality is no longer useful to classify irreps and, in particular, the fundamental irrep is no longer purely chiral or anti-chiral. We have developed techniques through dimensional reduction which allow us to obtain such irreps with more than one central charge. They will be described elsewhere. A similar situation arises for $N=4$, 6 and 8 supergravity where appropriate central charge irreps are also needed, as well as the irreps without central charges but with higher $Y$ and SU($N$) content. We will also report on this elsewhere.

Besides these applications to constructing linearised extended SYM and SGRS, we are interested in the possibility of extending certain of these irreps to be local representations. This will then allow for nonlinear supergravities to be constructed more directly.

Finally, we remark that these representations for $N>8$ are of interest in the construction of higher-spin interacting field theories. The question here is whether it is possible to nonlinearise these theories by suitable Noether techniques. It is usually said that such theories would be internally inconsistent. It is possible that such representations may avoid these difficulties, though further work would be needed to show this.

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